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# DYNAMIC RESPONSE OF A STRUCTURAL PANEL BY BOLOTIN'S METHOD†

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**Abstract**—Bolotin's asymptotic method is adopted for the investigation of dynamic response of a rectangular structural panel with elastic edge constraints resembling a box structure. Experimental determination on the frequency response is also included for comparison purpose. The method is proven to be extremely versatile in solving a broad class of the aforementioned problems.

## 1. INTRODUCTION

This study was motivated by the investigation of the dynamic response of a thin-walled rectangular box which has wide applications in many industrial problems. Instrumentation cabinets, transformer tanks and gear box casings are just a few of many possible examples. Hooker and O'Brien[1] used a finite-element method for the determination of natural frequency and mode shapes for a closed steel box which was previously determined by Dickinson and Warburton[2] who also investigated the natural frequencies of plate systems using the edge effect method[3].

In studying the dynamic response of a thin-walled rectangular box structure subjected to an excitation, the problem frequently can be reduced to the determination of the response of each of the wall panels if the wall thickness is small in comparison with the cross-sectional dimensions. By neglecting any coupling effects, the panels can be treated analytically by considering each one as a plate with different boundary conditions. For example, panel 1 as shown in Fig. 1, may be modeled as shown in Fig. 2, but with compliant supports at $y = 0$ and $y = b$, which include moment-resisting and deflection resisting springs. The stiffness of these springs generally varies with $x$ and $y$.

In solving such a problem, normally an approximate approach such as an energy method must be employed. If one is interested in not only the fundamental frequency of vibration, but also the response of higher modes, the results obtained by an energy approach become less reliable as the number of modes goes higher. In contrast to this drawback, the asymptotic

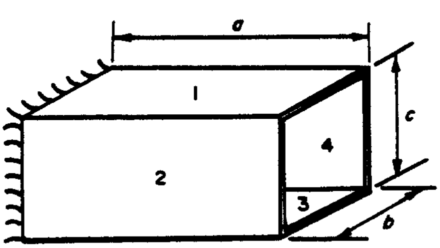


Fig. 1.

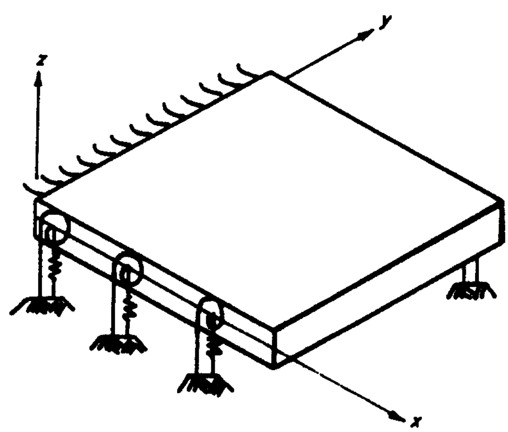


Fig. 2.

†An abstract of this paper was presented at CANCAM 75 (see Ref. [5]).

method due to Bolotin[4], can provide more accurate response results with very limited computing work involved. This can be observed from the problem solved and presented in this paper.

The method employed here is also often called the dynamic edge effect method, which is capable of finding the eigenvalues and eigenfunctions for one class of homogeneous linear boundary value problems in a rectangular region. According to this method, the asymptotic solution for eigenfunctions is expressed as the sum of a generating (or interior) solution and a corrective solution which is also called the "dynamic edge effect". The generating solution, expressed as a product of trignometric functions, satisfies the governing equation, but in general, does not satisfy the boundary conditions. The eigenvalues are determined from an algebraic equation and expressed as a function of "wave numbers". For each subregion, one constructs an asymptotic solution satisfying the governing equation and the conditions on the boundary. The number of these solutions is equal to the number of subregions. By joining these solutions together, one obtains an asymptotic solution for an eigenfunction of the entire region. As one moves toward the internal region, all these solutions tend to the generating solution if the dynamic edge effect is nondegenerate, i.e. the corrective solution is negligible in the internal area.

In this paper, the Bolotin's asymptotic method is adopted for the investigation of dynamic response of a structural panel. Experimental determination on the frequency response is also carried out for comparison purposes.

## 2. FORMULATION OF THE PROBLEM

The governing equation of free motion of a thin, isotropic, elastic plate is

$$\nabla^2\nabla^2 w + \frac{m}{D} w_{,tt} = 0 \tag{1}$$

where $\nabla^2$ is the two-dimensional Laplacian operator, $w = w(x,y,t)$, $m$ is the mass per unit area, $D$ is the plate flexural rigidity, and comma denotes the partial differentation.

For natural vibrations, one may set

$$w(x,y,t) = W(x,y) \sin \omega t \tag{2}$$

where $\omega$ is the circular frequency of vibration. Substitution of eqn (2) into eqn (1) gives

$$\nabla^2\nabla^2 W = \frac{m\omega^2}{D} W. \tag{3}$$

Along an elastically supported edge, e.g. $y = 0$ as shown in Fig. 2, the spring stiffness usually vary with $x$. In order to simplify the analysis, these springs are considered as constant along the length of the boundary. Then the shear force and bending moment along this edge can be related to the deflection and slope through the translational and rotational spring constants $r$ and $z$, respectively, i.e. along $x = 0$,

$$V(0) = r\,W(0) \tag{4}$$

$$M(0) = z\,W'(0) \tag{5}$$

Substituting for the shear force and bending moment,

$$EIW'''(0) = r\,W(0) \tag{6}$$

$$EIW''(0) = -z\,W'(0). \tag{7}$$

Since the stiffnesses are considered constant, and writing

$$Z = z/EI, \text{ and } R = r/EI$$

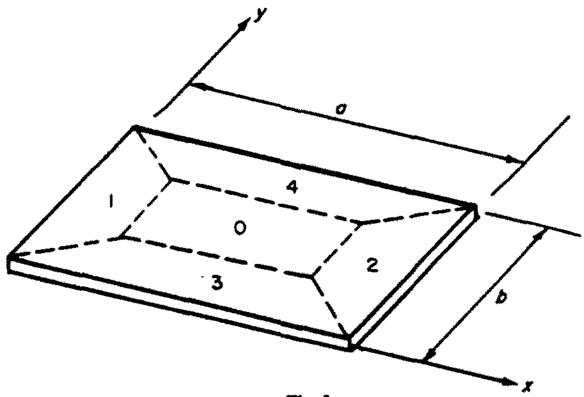


Fig. 3.

then the boundary conditions at $y = 0$ are

$$W'' = -Z_0 W' \tag{8}$$

$$W''' = R_0 W \tag{9}$$

while at $y = b$ the boundary conditions are

$$W'' = Z_B W' \tag{10}$$

$$W''' = R_B W. \tag{11}$$

Several limiting cases of interest can be examined by limiting constants $Z_0$, $Z_B$ and $R_0$, $R_B$. If both $Z_0 \to \infty$ and $Z_B \to \infty$ while $R_0 \to \infty$ and $R_B \to \infty$, the boundary conditions along the edges become those of clamped edges. Similarly, letting $R_0 \to \infty$ and $R_B \to \infty$ while $Z_0 \to 0$ and $Z_B \to 0$, the edges are simply supported.

The rectangular plate under consideration is divided into regions as shown in Fig. 3. Region (0) is considered free—sufficiently far from the boundaries that it is not influenced by any stiffening of the plate along the boundaries. Following Bolotin's method, the solution of the equilibrium equation for this region is assumed to be

$$W(x,y) = A[\sin(k_1x) + B\cos(k_1x)][\sin(k_2y) + C\cos(k_2y)]. \tag{12}$$

In the boundary regions (1–4) the solutions are those which satisfy the boundary conditions (except in the neighborhood of corners) and asymptotically approach the interior region solution at the intersection of the region boundaries. The solutions for each of the regions (1–4) are respectively

$$W_1(x,y) = \{D_1 \exp[-(k_1^2 + 2k_2^2)^{1/2}x] + A_1[\sin(k_1x) + B_2\cos(k_1x)]\}[\sin(k_2y) + C\cos(k_2y)] \tag{13}$$

$$W_2(x,y) = \{D_2 \exp[-(k_1^2 + 2k_2^2)^{1/2}(a - x)] + A_2[\sin(k_1x)] + B_2\cos(k_1x)]\}[\sin(k_2y) + C\cos(k_2y)] \tag{14}$$

$$W_3(x,y) = \{D_3 \exp[-(k_2^2 + 2k_1^2)^{1/2}y] + A_3[\sin(k_2y) + C_3\cos(k_2y)]\}[\sin(k_1y) + B\cos(k_1x)] \tag{15}$$

$$W_4(x,y) = \{D_4 \exp[-(k_2^2 + 2k_1^2)^{1/2}(b - y)] + A_4[\sin(k_2y) + C_4\cos(k_2y)]\}[\sin(k_1x) + B\cos k_1x)]. \tag{16}$$

Constants $k_1$ and $k_2$ are the spatial frequencies or wave numbers for the natural frequencies of the plate. Since these boundary regions must converge to the interior region solution then

$$A = A_1 = A_2 = A_3 = A_4 \tag{17}$$

$$B = B_1 = B_2 \tag{18}$$

$$C = C_3 = C_4. \tag{19}$$

Also, let $AB \equiv F_1$ and $AC \equiv F_2$, then invoking two boundary conditions for each of the boundaries at $x = 0$ and $x = a$, and substituting into eqns (13) and (14) respectively, produces four homogeneous equations in $A$, $F_1$, $D_1$ and $D_2$. For a non-trivial solution it is required that the determinant of the coefficients to be zero. Expanding the determinant yields a characteristic equation for vibrations propagating in the $x$-direction. This equation is transcendental in $k_1$ and $k_2$. Similarly, invoking boundary conditions at $y = 0$ and $y = b$ produces a second transcendental equation in $k_1$ and $k_2$. Solving the two characteristic equations simultaneously gives eigenvalues for $k_1$ and $k_2$. The natural frequencies of the plate are given by

$$\omega = \sqrt{\left(\frac{D}{m}\right)}(k_1^2 + k_2^2). \tag{20}$$

## 3. CHARACTERISTIC EQUATIONS AND THEIR SOLUTIONS

Recalling the model of a panel of a box structure with a cantilevered base and an open end (Fig. 2), the natural frequencies of oscillation can be determined using the boundary conditions for the compliant edges.

First, consider the cantilever boundary, $x = 0$. The boundary conditions are

$$W_1(0,y) = 0 \tag{21}$$

$$W_{1,x}(0,y) = 0 \tag{22}$$

where $W_1$ refers to (13).

The boundary at $x = a$ is free, it cannot support a shear force or bending moment, i.e.

$$W_{2,xx}(a,y) = 0 \tag{23}$$

$$W_{2,xxx}(a,y) = 0. \tag{24}$$

Substituting (13) and (14) into these boundary conditions yields four equations in $D_1$, $D_2$, $A$ and $F_1$ as follows:

$$D_1 + F_1 = 0 \tag{25}$$

$$-(k_1^2 + 2k_2^2)^{1/2} D_1 + k_1 A = 0 \tag{26}$$

$$(k_1^2 + 2k_2^2) D_2 - k_1^2 \sin(k_1 a) A - k_1^2 \cos(k_1 a) F_1 = 0 \tag{27}$$

$$(k_1^2 + 2k_2^2)^{3/2} D_2 - k_1^3 \cos(k_1 a) A + k_1^3 \sin(k_1 a) F_1 = 0. \tag{28}$$

Call the determinant of the coefficients of these four equations by $D_x$. Expanding this determinant and setting equal to zero, one has the characteristic equation

$$\tan(k_1 a) = \frac{k_1(k_1^2 + 2k_2^2)^{1/2}}{k_2^2}. \tag{29}$$

Considering now the boundary conditions along the boundaries with elastic constraints,

$y = 0$ and $y = b$, let $Z_0 = Z_B = Z$ and $R_0 = R_B = R$ for simplification purposes, then

$$W_{3,yy}(x,0) = -ZW_{3,y}(x,0) \tag{30}$$

$$W_{3,yyy}(x,0) = RW_3(x,0) \tag{31}$$

$$W_{4,yy}(x,b) = ZW_{4,y}(x,b) \tag{32}$$

$$W_{4,yyy}(x,b) = RW_4(x,b). \tag{33}$$

Substituting (15) and (16) into these boundary conditions yields four equations in $D_3$, $D_4$, $A$ and $F_2$. These are

$$[(k_2^2 + 2k_1^2) - Z(k_2^2 + 2k_1^2)^{1/2}]\, D_3 + Zk_2A - k_2^2F_2 = 0 \tag{34}$$

$$[R + (k_2^2 + 2k_1^2)^{3/2}]\, D_3 + k_2^3A + RF_2 = 0 \tag{35}$$

$$[(k_2^2 + 2k_1^2) - Z(k_2^2 + 2k_1^2)^{1/2}]\, D_4 + [Zk_2 \cos(k_2b) - k_2^2 \sin(k_2b)]\, A - [Zk_2 \sin(k_2b) + k_2^2 \cos(k_2b)]\, F_2 = 0 \tag{36}$$

$$[(k_2^2 + 2k_1^2)^{3/2} - R]\, D_4 - [k_2^3 \cos(k_2b) + R \sin(k_2b)]\, A + [k_2^3 \sin(k_2b) - R \cos(k_2b)]\, F_2 = 0. \tag{37}$$

The determinant of the coefficients of these four equations is tedious to evaluate. Instead, by constraining $R$ and $Z$, characteristic equations can be derived for different boundary conditions of interest. Letting $R \to \infty$, and letting $Z \to 0$, the boundaries are simply supported. For this case, the characteristic equation obtained by setting the determinant equal to zero is

$$\sin(k_2b) = 0. \tag{38}$$

To produce clamped boundaries, let both $R \to \infty$ and $Z \to \infty$. The characteristic equation is

$$\tan(k_2b) = \frac{k_2(k_2^2 + 2k_1^2)^{1/2}}{k_1^2}. \tag{39}$$

Eigenvalues for $k_1a$ and $k_2b$ are obtained by solving eqns (29) and (39) or eqns (29) and (38) simultaneously. The eigenvalues obtained will be dependent on the ratio of plate dimensions $a/b$. As an example, first consider a plate with cantilever–cantilever–free–cantilever boundaries. Using iteration scheme, $k_1a$ and $k_2b$ are determined (Table 1). Once the plate dimensions are given, then the spatial frequencies $k_1$ and $k_2$ are computed (Table 2).

For plates made from 0.060 in. (1.5 mm) thick aluminum, one has

$$E = 10.0 \times 10^6 \text{psi} = 68.95\ \text{GN/m}^2$$

$$\nu = 0.32$$

$$\rho = 0.097\ \text{lb/in}^3 = 2710\ \text{kg/m}^3.$$

Table 1. Eigenvalues for rectangular plates with C–C–F–C boundaries.

| $k_1a$ | | $k_2b$ | |
|---|---|---|---|
| a/b=2.0 | a/b=1.5 | a/b=2.0 | a/b=1.5 |
| 1.053 | 0.790 | 4.700 | 4.706 |
| 5.519 | 5.346 | 4.443 | 4.293 |
| 7.234 | 7.442 | 4.257 | 4.055 |
| 11.320 | 11.190 | 3.904 | 3.700 |
| 13.890 | 14.010 | 3.899 | 3.606 |

Table 2. Spatial frequencies for rectangular plates with C–C–F–C boundaries*

| $k_1$ | | $k_2$ | | $k_1^2 + k_2^2$ | |
|---|---|---|---|---|---|
| 0.088 | 0.066 | 0.783 | 0.783 | 0.621 | 0.617 |
| 0.460 | 0.446 | 0.740 | 0.716 | 0.757 | 0.712 |
| 0.603 | 0.620 | 0.710 | 0.675 | 0.868 | 0.841 |
| 0.943 | 0.933 | 0.651 | 0.617 | 1.313 | 1.251 |
| 1.158 | 1.168 | 0.650 | 0.601 | 1.764 | 1.724 |

*The first column under each heading is for a=12" (305mm) and b=6" (152mm), while the second column is for a=12" (305mm) and b=8" (203mm).

Substituting these and the values from Table 2 into eqn (20) gives frequencies as shown in Table 3.

## 4. EXPERIMENTAL WORK

The purpose of the experimental work was to obtain the dynamic response data and to compare the results to those predicted by analytic work. The natural frequencies of the structure were determined and the mode shapes were observed.

A thin-wall rectangular box structure with an open end and a cantilevered base, dimensions 12×8×6 in. (305×203×152 mm), was constructed from a sheet of 0.06 in. (1.5 mm) thick aluminum plate. The box structure was mounted in a shaker table, MB Electronics Model EA2150 Vibration Exciter, with a mounting bracket fixed to the contilevered base. The shaker table was then driven by the exciter control, and the frequency scan of the exciter control engaged. A frequency range of 5–5000 Hz was scanned. The resonant frequencies of a panel on the box structure were determined by spreading a thin layer of salt on the panel and adjusting the frequency frequency control until the formation of nodal lines on the panel was observed.

The first five natural frequencies of a 12×6 in. panel on the box structure and the corresponding mode shapes were determined (Fig. 4). The analytically and experimentally determined natural frequencies are compared in Table 4.

In order to compare experimental and analytic results, the effect of all assumptions made in experimentation must be assessed. The first assumption made was that the edge constraints on the model of the box structure were very stiff—the panel on the box could be considered as having cantilever–cantilever–free–cantilever boundaries. This consideration would indicate that

Table 3. Frequency, Hz

| | 12" x 6" plate (305mm x 152mm) | 12" x 8" plate (305mm x 203mm) |
|---|---|---|
| $f_1$ | 360 | 358 |
| $f_2$ | 440 | 413 |
| $f_3$ | 503 | 488 |
| $f_4$ | 761 | 725 |
| $f_5$ | 1023 | 1000 |

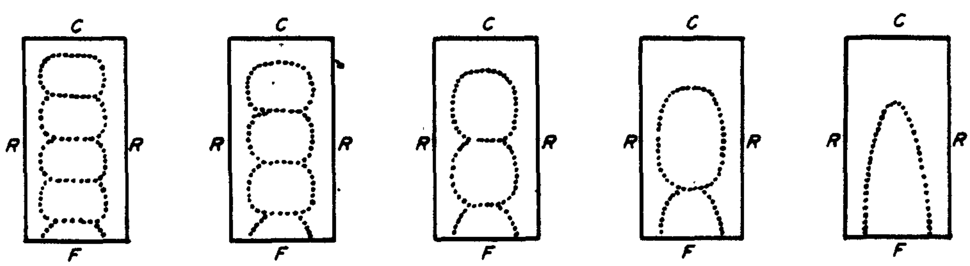


Fig. 4.

Table 4. Comparison of calculated and experimentally determined natural frequencies

| | Calculated Frequency (Hz) | Measured Frequency (Hz) | Difference % |
|---|---|---|---|
| $f_1$ | 360 | 290 | 24% |
| $f_2$ | 440 | 360 | 22% |
| $f_3$ | 503 | 510 | 1.4% |
| $f_4$ | 761 | 750 | 1.4% |
| $f_5$ | 1023 | 1090 | 6.0% |

the natural frequencies calculated for the idealized panel would be slightly higher than the analytic results of the actual panel since the latter has some compliancy at the boundaries. As the torsional stiffness of the boundary varies from zero to infinite, the natural frequencies increase.

Coupling effects induced by structure panels with adjacent boundaries, assumed negligible, and errors introduced by the asymptotic approximation, both discussed in the next section, are sources of analytic discrepancy with experimental work.

The resonant frequencies determined experimentally were not the natural frequencies of free vibration, but the frequencies of forced vibration. However, from the sharpness of the resonances observed, it was concluded that the natural frequencies of free vibration occurred at the resonant frequencies.

A final assumption made in the experimental work was that the mounting bracket was sufficiently rigid that it did not introduce any coupling effects between the box structure and the shaker table.

## 5. DISCUSSIONS AND CONCLUSIONS

The close correlation between the calculated natural frequencies of a panel of a thin-walled box structure and the experimentally determined natural frequencies proves the merit of Bolotin's asymptotic method, despite the fact that certain deviations exist in the first two modes. The Rayleigh–Ritz method, which would provide a more accurate determination for the fundamental frequency, would be unable to determine the higher natural frequencies with such accuracy.

The reason for the inaccuracy of the asymptotic method at the lower frequencies can be seen in the model of the plate with interior and boundary regions as shown in Fig. 3. One assumption of the asymptotic solution is that the interior region (region 0) is assumed to be free from the influence of the stiffening effects of the constrained boundaries. This assumption poses a question of degree. For very large plates, the edge stiffening effects are minimal at the interior, but for small plates, the interior region is affected by the stiffened boundaries.

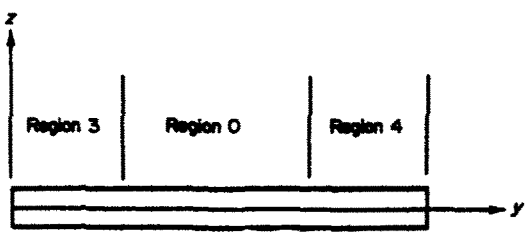


Fig. 5.

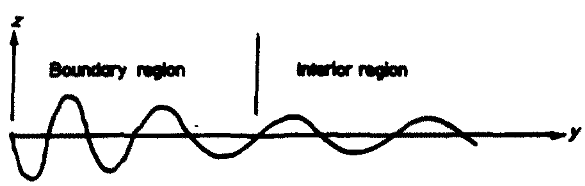


Fig. 6.

Figure 5 shows a plane cutting the interior region parallel to the $y$–$z$ plane. From this cross section, waves propagating in the $y$-direction can be examined. The Bolotin solution assumes that the waves in regions 3 and 4 decay exponentially to match waves in region 0 at the region divisions. For this to occur, however, the length of the waves in each region must be much shorter than the width of the region (Fig. 6). Since this does not occur in the range of low natural frequency, it therefore introduces a significant error in the determination of the lower natural frequencies. For example, from eqn (13), it is seen that the exponential term at a specific value of $x$ decreases as the frequency $\omega$ (eqn 20) increases. Hence the errors decrease with increase in frequency.

In conclusion, it may be stated that the asymptotic method has been proven extremely versatile in solving a broad class of problems dealing with the determination of natural frequencies of vibration of rectangular panels with elastic edge constraints. Its use in studying the dynamic response of box-type structure is limited only by its inability to accurately determine the fundamental frequency, and the complexity of the characteristic equations involved.

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